# Time Series: M.Phil. Econometrics, Hilary Term 2014
# Solutions for week 6 problem set

Oleg I. Kitov

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## Useful concepts

**Definition 1.** (Convergence in distribution) A sequence $X_1, X_2, \ldots$ of random variables with cumulative distribution functions $F_n(x)$ for $x \in \mathbb{R}$ is said to converge in distribution to a random variable $X$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for every $x \in \mathbb{R}$. We say that $X_n \xrightarrow{\mathsf{D}} X$.

**Definition 2.** (Convergence in probability) A sequence $X_1, X_2, \ldots$ of random variables said to converge in probability to a random variable $X$ if $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \mathsf{P}(|X_n - X| \geq \varepsilon) = 0$$

We can write this as $X_n \xrightarrow{\mathsf{P}} X$. In other words, for any $\varepsilon > 0$ and $\delta > 0$ we can always find some $N$ such that for all $n > N$ the probability $\mathsf{P}(|X_n - X| \geq \varepsilon) < \delta$. Note that convergence in probability implies convergence in distribution.

## Question 1

Consider data $X_{-1}, X_0, X_1, \ldots X_T$ and the model

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \epsilon_t, \quad \text{for } t = 1, ..., T, \tag{1}$$

conditionally on $X_0, X_{-1}$, where $\epsilon_t$ are independent $N\left(0, \sigma^2\right)$.

(a) Rewrite the model equation in equilibrium correction form

$$\Delta X_t = \pi X_{t-1} + \gamma \Delta X_{t-1} + \epsilon_t \tag{2}$$

What is the connection between $(\pi, \gamma)$ and $(\alpha_1, \alpha_2)$?

This representation is very common, since we link the change in the variable to its previous realization. This is also a way to transform the relationship into a stationary representation in case $X_t$ has a unit root - we will see this in part (b). Mechanically, the idea here is to subtract $X_{t-1}$ from both sides of the original representation and add and subtract $\alpha_2 X_{t-1}$ on the left hand side. We get:

$$\begin{aligned}
X_t &= \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \epsilon_t & (3)\\
X_t - X_{t-1} &= \alpha_1 X_{t-1} - X_{t-1} + \alpha_2 X_{t-1} - \alpha_2 X_{t-1} + \alpha_2 X_{t-2} + \epsilon_t & (4)\\
\Delta X_t &= (\alpha_1 + \alpha_2 - 1) X_{t-1} - \alpha_2 \Delta X_{t-1} + \epsilon_t & (5)\\
\Delta X_t &= \pi X_{t-1} + \gamma \Delta X_{t-1} + \epsilon_t & (6)
\end{aligned}$$

so that $\pi = (\alpha_1 + \alpha_2 - 1)$ and $\gamma = -\alpha_2$.

(b) Recall from §7.2.5 that $X_t$has at least one unit root when $\alpha_1 + \alpha_2 = 1$ and two unit roots if also $\alpha_1 = 2$. What do those statement imply about the representation $(\pi, \gamma)$ and the equation (2).

$\alpha_1 + \alpha_2 = 1$ implies $\pi = 0$. If also $\alpha_1 = 2$ then also $\gamma = 1$. What does that imply for the process of $X_t$?

- If $\pi = 0$ then $\Delta X_t = \gamma \Delta X_{t-1} + \epsilon_t$. That is the first difference is stationary.
- If $\pi = 0$ and $\gamma = 1$ then $\Delta^2 X_t = \epsilon_t$. That is the second difference is stationary.

(c) Derive the likelihood for the model parametrized in (2)

This is something that we have been doing for the last three weeks, so we just do this once again with a slightly different notation and make the substitution of the ADL form. The joint density for $X_1, \ldots, X_T$ given $X_0, X_{-1}$ is

$$\begin{aligned}
& \mathsf{f}_{\alpha_1,\alpha_2,\sigma^2}(x_1, \ldots, x_T | x_0, x_{-1}) && (7)\\
= & \text{[ prediction decomposition ]} = \prod_{t=1}^{T} \mathsf{f}_{\alpha_1,\alpha_2,\sigma^2} && (8)\\
= & \text{[ 2nd order proces ]} = \prod_{t=1}^{T} \mathsf{f}_{\alpha_1,\alpha_2,\sigma^2}(x_t | x_{t-1}, x_{t-2}) && (9)\\
= & \text{[ normality ]} = \prod_{t=1}^{T} (2\pi\sigma^2)^{-1/2} \exp\left\{ -\frac{1}{2\sigma^2}(x_t - \alpha_1 x_{t-1} - \alpha_2 x_{t-2})^2 \right\} && (10)\\
= & \text{[ using ADL ]} = \prod_{t=1}^{T} (2\pi\sigma^2)^{-1/2} \exp\left\{ -\frac{1}{2\sigma^2}(\Delta x_t - \pi x_{t-1} - \gamma \Delta x_{t-1})^2 \right\} && (11)\\
= & (2\pi\sigma^2)^{-T/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{t=1}^{T} (\Delta x_t - \pi x_{t-1} - \gamma \Delta x_{t-1})^2 \right\} && (12)
\end{aligned}$$

leading to log likelihood

$$\ell_{X_1,\ldots,X_T|X_0,X_{-1}}(\alpha_1, \alpha_2, \sigma^2) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}(\Delta X_t - \pi X_{t-1} - \gamma \Delta X_{t-1})^2 \quad (13)$$

(d) Derive the maximum likelihood estimator of $\pi$.

By now, we all know very well how to do this (either directly or using orthogonolisation), so I will just present the final result here. Make sure are familiar with this an know how to derive it (if unsure take a look at question 1 from weeks 3 and 4).

$$\widehat{\pi} = \frac{\sum_{t=1}^{T} \Delta X_t (X_{t-1} | \Delta X_{t-1})}{\sum_{t=1}^{T} (X_{t-1} | \Delta X_{t-1})^2} = \frac{\sum_{t=1}^{T} \Delta X_t \left( X_{t-1} - \frac{\sum_{s=1}^{T} X_{s-1} \Delta X_{s-1}}{\sum_{s=1}^{T} (\Delta X_{s-1})^2} \Delta X_{t-1} \right)}{\sum_{t=1}^{T} \left( X_{t-1} - \frac{\sum_{s=1}^{T} X_{s-1} \Delta X_{s-1}}{\sum_{s=1}^{T} (\Delta X_{s-1})^2} \Delta X_{t-1} \right)^2} \quad (14)$$

(e) Is there any relation between $\hat{\pi}$ and the maximum likelihood estimators for $\alpha_1, \alpha_2$?

Since $\pi = \alpha_1 + \alpha_2 - 1$ then $\widehat{\pi} = \widehat{\alpha}_1 + \widehat{\alpha}_2 - 1$. This follows from the equivariance of likelihoods, see §2.1.5 in notes from Michaelmas.

## Question 2

Please see Bent Nielsen's solutions, the Ox Script and the corresponding .out file.

## Question 3

Please see Bent Nielsen's solutions, the Ox Script and the corresponding .out file.

## Question 4

Consider data $Y_0, Y_1, \ldots, Y_T$ and the model

$$Y_t = \alpha Y_{t-1} + \mu + \epsilon_t \text{ for } t = 1, \ldots, T. \tag{15}$$

The least squares estimator for $\mu$ equals

$$\hat{\mu} = \frac{\sum_{t=1}^{T} Y_t \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} \tag{16}$$

Find the asymptotic distribution of $\hat{\mu}$ when $|\alpha| < 1$ in the following steps.

(a) Argue that $Y_t = X_t + \mu_Y$ where $X_t = \alpha X_{t-1} + \epsilon_t$.

Again, we have seen a similar derivation before. Let's proceed by evaluation the relationship between $\mu$ and $\mu_Y$ by subtracting the long-term mean of $Y_t$, $\mu_Y = \mu / (1 - \alpha)$.

$$Y_t - \mu_Y = \alpha Y_{t-1} + \mu - \mu_Y + \epsilon_t \tag{17}$$

$$Y_t - \mu_Y = \alpha Y_{t-1} + \mu_Y (1 - \alpha) - \mu_Y + \epsilon_t \tag{18}$$

$$Y_t - \mu_Y = \alpha (Y_{t-1} - \mu_Y) + \epsilon_t \tag{19}$$

$$X_t = \alpha X_t + \epsilon_t \tag{20}$$

This is a regression for the demeaned series.

(b) Start by evaluating the estimate of $\mu$ by substituting in the equation for $Y_t$

$$\hat{\mu} = \frac{\sum_{t=1}^{T} Y_t \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} \tag{21}$$

$$= \frac{\sum_{t=1}^{T} (\alpha Y_{t-1} + \mu + \epsilon_t) \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} \tag{22}$$

$$= \frac{\sum_{t=1}^{T} (\alpha Y_{t-1} + \mu) \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} + \frac{\sum_{t=1}^{T} \epsilon_t \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} \tag{23}$$

First, we not that as the question asks we can define:

$$N = \sum_{t=1}^{T} \epsilon_t \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right), \quad D = \sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2 \tag{24}$$

Now, we only have to simplify the first part of this equation. Note that the letter we use in the indexing does not matter for the total summation, that is $\sum_{s=1}^{T} Y_{s-1} = \sum_{t=1}^{T} Y_{t-1}$. Also, summations over $s$ are constant in relation to summation of $t$, so we can factor them out in what will follow. Bearing that in mind we can evaluate this expression in the following way:

$$= \frac{\sum_{t=1}^{T} (\alpha Y_{t-1} + \mu) \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2} \tag{25}$$

$$= \frac{\sum_{t=1}^{T} \left(\alpha Y_{t-1} - \alpha \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}^2 + \mu - \mu \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)}{\sum_{t=1}^{T} \left(1 - 2\frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1} + \left(\frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} Y_{t-1}\right)^2\right)} = \tag{26}$$

$$= \frac{\alpha \sum_{t=1}^{T} Y_{t-1} - \alpha \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} \sum_{t=1}^{T} Y_{t-1}^2 + \sum_{t=1}^{T} \mu - \mu \frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} \sum_{t=1}^{T} Y_{t-1}}{\sum_{t=1}^{T} 1 - 2\frac{\sum_{s=1}^{T} Y_{s-1}}{\sum_{s=1}^{T} Y_{s-1}^2} \sum_{t=1}^{T} Y_{t-1} + \frac{\left(\sum_{s=1}^{T} Y_{s-1}\right)^2}{\left(\sum_{s=1}^{T} Y_{s-1}^2\right)^2} \sum_{t=1}^{T} Y_{t-1}^2} \tag{27}$$

$$= \frac{\mu \left(\sum_{t=1}^{T} 1 - \frac{\left(\sum_{s=1}^{T} Y_{s-1}\right)^2}{\sum_{s=1}^{T} Y_{s-1}^2}\right)}{\left(\sum_{t=1}^{T} 1 - \frac{\left(\sum_{s=1}^{T} Y_{s-1}\right)^2}{\sum_{s=1}^{T} Y_{s-1}^2}\right)} \tag{28}$$

$$= \mu \tag{29}$$

(c) We have just done that above, when evaluating the denominator.

(d) This is a direct manipulation of the expression for $D$ and substituting in the expression for $Y_t = X_t + \mu_Y$

$$T^{-1}D = T^{-1}\left(\sum_{t=1}^{T} 1 - \frac{\left(\sum_{s=1}^{T} Y_{s-1}\right)^2}{\sum_{s=1}^{T} Y_{s-1}^2}\right) \tag{30}$$

$$= T^{-1}\sum_{t=1}^{T} 1 - \frac{T^{-2}\left(\sum_{s=1}^{T} Y_{s-1}\right)^2}{T^{-1}\sum_{s=1}^{T} Y_{s-1}^2} \tag{31}$$

$$= T^{-1}\sum_{t=1}^{T} 1 - \frac{\left(T^{-1}\sum_{s=1}^{T} Y_{s-1}\right)^2}{T^{-1}\sum_{s=1}^{T} Y_{s-1}^2} \tag{32}$$

$$= T^{-1}\sum_{t=1}^{T} 1 - \frac{\left(T^{-1}\sum_{s=1}^{T} (X_{s-1}+\mu_Y)\right)^2}{T^{-1}\sum_{s=1}^{T} (X_{s-1}+\mu_Y)^2} \tag{33}$$

(e) Recall the results from Theorem 4.5 for stationary processes, with *iid* residuals and $T \to \infty$.

$$T^{-1}\sum_{t=1}^{T} X_{t-j} \xrightarrow{\mathsf{P}} 0, \text{ for } j = 0,1 \tag{34}$$

$$T^{-1}\sum_{t=1}^{T} X_{t-j}^2 \xrightarrow{\mathsf{P}} \frac{\sigma^2}{1-\alpha^2} = \sigma_X^2, \text{ for } j = 0,1 \tag{35}$$

$$T^{-1}\sum_{t=1}^{T} \epsilon_t X_{t-1} \xrightarrow{\mathsf{P}} 0 \tag{36}$$

Theorem 4.6 states the following results for the convergence in distribution

$$T^{-1/2}\sum_{t=1}^{T} X_{t-j} \xrightarrow{\mathsf{D}} \mathsf{N}\left(0, \frac{\sigma^2}{1-\alpha^2}\right) = \mathsf{N}\left(0, \sigma_X^2\right), \text{ for } j = 0,1 \tag{37}$$

$$T^{-1/2}\sum_{t=1}^{T} \epsilon_t X_{t-1} \xrightarrow{\mathsf{D}} \mathsf{N}\left(0, \frac{\sigma^4}{1-\alpha^2}\right) = \mathsf{N}\left(0, \sigma^2\sigma_X^2\right) \tag{38}$$

Now consider the expression for $T^{-1}D$ term-by-term:

- The first term:

$$T^{-1}\sum_{t=1}^{T} 1 = 1$$

- The upper part of the second term: using continuous mapping theorem:

$$\left(T^{-1}\sum_{s=1}^{T} (X_{s-1}+\mu_Y)\right)^2 = \left(T^{-1}\sum_{s=1}^{T} X_{s-1}+\mu_Y\right)^2 \xrightarrow{\mathsf{P}} \mu_Y^2$$

- The lower part of the second term:

$$T^{-1}\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2 = T^{-1}\sum_{s=1}^{T}\left(X_{s-1}^2+2X_{s-1}\mu_Y+\mu_Y^2\right) \xrightarrow{\mathsf{P}} \sigma_Y^2+\mu_Y^2$$

- Combining terms

$$T^{-1}D \xrightarrow{\mathsf{P}} 1-\frac{\mu_Y^2}{\sigma_Y^2+\mu_Y^2} = \frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2} \tag{39}$$

(f) Substitute $Y_t = X_t + \mu_Y$ into the expression for $N$:

$$\begin{aligned} N &= \sum_{t=1}^{T}\epsilon_t\left(1-\frac{\sum_{s=1}^{T}Y_{s-1}}{\sum_{s=1}^{T}Y_{s-1}^2}Y_{t-1}\right) && (40)\\ &= \sum_{t=1}^{T}\epsilon_t\left(1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}(X_{t-1}+\mu_Y)\right) && (41)\end{aligned}$$

(g)

$$\begin{aligned} T^{-1/2}N &= T^{-1/2}\sum_{t=1}^{T}\epsilon_t\left(1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}(X_{t-1}+\mu_Y)\right) && (42)\\ &= T^{-1/2}\sum_{t=1}^{T}\epsilon_t\left(-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}X_{t-1}+1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}\mu_Y\right)\\ &= -T^{-1/2}\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}\sum_{t=1}^{T}X_{t-1}\epsilon_t+T^{-1/2}\left(1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}\mu_Y\right)\sum_{t=1}^{T}\epsilon_t\\ &= \left(-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}, 1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}\mu_Y\right)T^{-1/2}\sum_{t=1}^{T}\begin{pmatrix}X_{t-1}\epsilon_t\\ \epsilon_t\end{pmatrix}\end{aligned}$$

where in the last equality we noted that the summation can be expressed in terms of two $2\times 1$ vectors.

(h) We need to find the limit of the big fraction in the first vector, and from the asymptotic results we mentioned above here is what follows:

$$\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2} = \frac{T^{-1}\sum_{s=1}^{T}X_{s-1}+\mu_Y}{T^{-1}\sum_{s=1}^{T}X_{s-1}^2+2\mu_Y T^{-1}\sum_{s=1}^{T}X_{s-1}+\mu_Y^2} \xrightarrow{\mathsf{P}} \frac{\mu_Y}{\sigma_Y^2+\mu_Y^2} \tag{43}$$

Hence, the probability limit of the first vector in (49) is:

$$\left(-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}, 1-\frac{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)}{\sum_{s=1}^{T}(X_{s-1}+\mu_Y)^2}\mu_Y\right) \xrightarrow{\mathsf{P}} \left(-\frac{\mu_Y}{\sigma_Y^2+\mu_Y^2}, \frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) \tag{44}$$

The distribution limit of the second vector is know directly form Theorems 4.5 and 4.6 and equation 4.32:

$$T^{-1/2}\sum_{t=1}^{T}\begin{pmatrix} X_{t-1}\epsilon_t \\ \epsilon_t \end{pmatrix} \xrightarrow{\mathsf{D}} \mathsf{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2\begin{pmatrix} \sigma_Y^2 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{45}$$

Now we can combine these asymptotic results and through some manipulation with the vector product we get the following:

$$\begin{aligned}
T^{-1/2}N &\xrightarrow{\mathsf{D}} \left(-\frac{\mu_Y}{\sigma_Y^2+\mu_Y^2}, \frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right)\mathsf{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma^2\begin{pmatrix} \sigma_Y^2 & 0 \\ 0 & 1 \end{pmatrix}\right) && (46)\\
&= \mathsf{N}\left(0, \left(\frac{\mu_Y}{\sigma_Y^2+\mu_Y^2}\right)^2\sigma^2\sigma_Y^2 + \sigma^2\left(\frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right)^2\right) && (47)\\
&= \mathsf{N}\left(0, \frac{\mu_Y^2\sigma^2\sigma_Y^2}{\left(\sigma_Y^2+\mu_Y^2\right)^2} + \frac{\sigma^2\sigma_Y^4}{\left(\sigma_Y^2+\mu_Y^2\right)^2}\right) && (48)\\
&= \mathsf{N}\left(0, \frac{\sigma^2\sigma_Y^2\left(\sigma_Y^2+\mu_Y^2\right)}{\left(\sigma_Y^2+\mu_Y^2\right)^2}\right) && (49)\\
&= \mathsf{N}\left(0, \frac{\sigma^2\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) && (50)
\end{aligned}$$

(i) Now we can combine the results from (e) and (h) to get the following:

$$\begin{aligned}
T^{1/2}\left(\hat{\mu}-\mu\right) &= T^{1/2}\frac{N}{D} && (51)\\
&= \frac{T^{-1/2}}{T^{-1}}\frac{N}{D} && (52)\\
&\xrightarrow{\mathsf{D}} \left(\frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right)^{-1}\mathsf{N}\left(0, \frac{\sigma^2\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) && (53)\\
&= \mathsf{N}\left(0, \left(\frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right)^{-2}\frac{\sigma^2\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) && (54)\\
&= \mathsf{N}\left(0, \frac{\left(\sigma_Y^2+\mu_Y^2\right)^2}{\sigma_Y^4}\frac{\sigma^2\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) && (55)\\
&= \mathsf{N}\left(0, \left(\sigma_Y^2+\mu_Y^2\right)\frac{\sigma^2}{\sigma_Y^2}\right) && (56)
\end{aligned}$$

(j) First, we replace $\hat{\mu} - 0 = N/D$ to get:

$$
\begin{aligned}
Z_{\mu=0} &= \frac{(\hat{\mu}-0)}{(s^2/D)^{1/2}} && (57)\\
&= \frac{N/D}{(s^2/D)^{1/2}} && (58)\\
&= \frac{N}{(s^2D)^{1/2}} && (59)\\
&= \frac{T^{-1/2}N}{(s^2T^{-1}D)^{1/2}} && (60)\\
&\overset{\mathsf{D}}{\to} \left(\sigma^2\frac{\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right)^{-1/2}\mathsf{N}\left(0,\frac{\sigma^2\sigma_Y^2}{\sigma_Y^2+\mu_Y^2}\right) && (61)\\
&= \mathsf{N}(0,1) && (62)
\end{aligned}
$$

(k) The asymptotic variance of $N$ matches the limit of $D$. Hence the t-statistic is asymptotically standard normal. This is different from static regression, where the regression does not match the data generating process.