Econometrics, Michaelmas Term 2015, Quiz 6

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1. Consider a (multiple) linear regression, $Y = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where X is an $n \times k$ matrix of explanatory variables, x_j=$(x_{j1}, x_{j2}, \ldots, x_{jn})$, $j = 1, \ldots, k$ and $\boldsymbol{\varepsilon}$ are *i.i.d.* error terms. Write down the Gauss-Markov Theorem for $\hat{\boldsymbol{\beta}}$, the OLS estimator of $\boldsymbol{\beta}$. Explicitly state and explain all underlying assumptions.

Theorem. ***(Gauss-Markov):*** *The OLS estimator $\hat{\boldsymbol{\beta}}$ is BLUE: (Best (most efficient), Linearly (conditionally) Unbiased Estimator), under the following assumptions:*

1. Linear in parameters: The relationship between Y and $x_1, x_2, \ldots, x_k$ is given by $Y = X\beta + \varepsilon$, i.e. $y_i = \sum_{j=1}^{k} \beta_j x_{ji} + \varepsilon_i$.
2. X is full rank: $rank\,(X) = k$, i.e. the explanatory variables are linearly independent.
3. $\mathrm{E}\left[\varepsilon_i | X\right] = 0$, $\forall i = 1, \ldots, n$: the error term is independent of the explanatory variables.
4. Spherical errors: $\mathrm{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\right] = \sigma^2 I_n$, i.e. the residuals are homoscedastic and uncorrelated.
5. Fixed or random regressors X.

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2. Show that the OLS estimator $\hat{\boldsymbol{\beta}}$ is (conditionally) unbiased for the true parameter $\boldsymbol{\beta}$.

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\begin{aligned}
\mathrm{E}\left[\hat{\boldsymbol{\beta}}|X\right] &= \mathrm{E}\left[\left(X'X\right)^{-1} X'Y|X\right] \\
\text{(substitute the model for } Y) &= \mathrm{E}\left[\left(X'X\right)^{-1} X'\left(X\boldsymbol{\beta} + \boldsymbol{\varepsilon}\right)|X\right] \\
&= \mathrm{E}\left[\left(X'X\right)^{-1} X'X\boldsymbol{\beta}|X\right] + \mathrm{E}\left[\left(X'X\right)^{-1} X'\boldsymbol{\varepsilon}|X\right] \\
&= \boldsymbol{\beta} + \left(X'X\right)^{-1} X'\mathrm{E}\left[\boldsymbol{\varepsilon}|X\right] \\
\text{(since } \mathrm{E}\left[\boldsymbol{\varepsilon}|X\right] = 0) &= \boldsymbol{\beta}
\end{aligned}
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3. State the Law of Large Numbers and the Central Limit Theorem for *i.i.d.* random variables X_i with mean μ and variance $\sigma^2 < \infty$. Explain in words what these results mean.

Theorem. ***(Law of Large Numbers):*** *If X_i are i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$, then $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$, as $n \to \infty$. Sample mean converges in probability to the population mean as the sample size tends to infinity. In other words, as the sample size tends to infinity, the probability that the sample mean will converge to the population mean converges to 1.*

Theorem. ***(Central Limit Theorem):*** *If X_i are i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$, then $\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)\right) \xrightarrow{d} \mathrm{N}\left(0, \sigma^2\right)$ or equivalently $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{d} \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$, as $n \to \infty$. As the sample size tends to infinity, the sample mean converges in distribution to a normal distribution with mean equal to the population mean of X_i and variance decreasing at the rate $\frac{1}{n}$.*

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4. State the Law of Large Numbers and the Central Limit Theorem for weakly (covariance) stationary random variables X_t with $\mathrm{E}\left[X_t\right] = \mu$ and absolutely summable autocovariances, i.e. $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, where $\gamma(h) = \mathrm{Cov}\left[X_t, X_{t-h}\right]$. Are these results different from the LLN and the CLT for *i.i.d.* random variables?

Theorem. ***(Law of Large Numbers):*** *If X_i are weakly stationary random variables, then $\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{p} \mu$, as $n \to \infty$.*

Theorem. ***(Central Limit Theorem):*** *If X_i are weakly stationary and admit an $MA(\infty)$ represention $X_t = \mu + \Psi(L)\,\varepsilon_t$, where $\varepsilon_t \sim i.i.d.\left(0, \sigma_\varepsilon^2\right)$, then $\sqrt{n}\left(\left(\frac{1}{n}\sum_{t=1}^{n} X_t - \mu\right)\right) \xrightarrow{d} \mathrm{N}\left(0, \sum_{h=-\infty}^{\infty}|\gamma(h)|\right)$, where $\sum_{h=-\infty}^{\infty}|\gamma(h)|$ is the long-run variance.*