Econometrics, Michaelmas Term 2015, Quiz 4(1)

Consider a linear cross-sectional regression, $Y = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where X is an $N \times K$ matrix of explanatory variables and $\boldsymbol{\varepsilon}$ is the error term. At this stage, no assumptions are made. **Write down explcitly any assumptions** that you make when answering the following questions:

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1. What are the dimensions of $Y, \boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$?

$$\underset{(N\times 1)}{Y} = \underset{(N\times K)}{X}\underset{(K\times 1)}{\boldsymbol{\beta}} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}},$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & \dots & x_{K1} \\ x_{12} & x_{22} & \dots & x_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1N} & x_{2N} & \dots & x_{KN} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix}$$

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2. Write down (but do not derive) the OLS estimator $\hat{\boldsymbol{\beta}}$ in matrix notation.

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1} X'Y$$

Assuming that X is full rank, i.e. $rank(X) = K$ and so $(X'X)$ is invertible.

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3. Show that $\hat{\boldsymbol{\beta}}$ is a (conditionally) unbiased estimator of the true parameter $\boldsymbol{\beta}$.

$$\begin{aligned} \mathrm{E}\left[\hat{\boldsymbol{\beta}}|X\right] &= \mathrm{E}\left[(X'X)^{-1} X'Y|X\right] \\ &= \mathrm{E}\left[(X'X)^{-1} X'(X\boldsymbol{\beta} + \boldsymbol{\varepsilon})|X\right] \\ &= \mathrm{E}\left[(X'X)^{-1} X'X\boldsymbol{\beta}|X\right] + \mathrm{E}\left[(X'X)^{-1} X'\boldsymbol{\varepsilon}|X\right] \\ &= \boldsymbol{\beta} + (X'X)^{-1} X'\mathrm{E}\left[\boldsymbol{\varepsilon}|X\right] \\ (\text{assume } \mathrm{E}\left[\boldsymbol{\varepsilon}|X\right] = 0) &= \boldsymbol{\beta} \end{aligned}$$

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4. Derive the expression for the conditional variance of $\hat{\boldsymbol{\beta}}$ if the error term is homoscedastic. Is this different from the unconditional variance of $\hat{\boldsymbol{\beta}}$?

First notice that $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (X'X)^{-1} X'\boldsymbol{\varepsilon}$ and that $\mathrm{Var}\left(\hat{\boldsymbol{\beta}}|X\right) = \mathrm{Var}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}|X\right)$. Assume $\mathrm{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|X\right] = \sigma_\varepsilon^2 I_n$. Then the conditional variance is given by:

$$\begin{aligned} \mathrm{Var}\left(\hat{\boldsymbol{\beta}}|X\right) &= \mathrm{E}\left[\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)'|X\right] \\ &= \mathrm{E}\left[(X'X)^{-1} X'\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'X(X'X)^{-1}|X\right] \\ &= (X'X)^{-1} X'\mathrm{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|X\right]X(X'X)^{-1} \\ &= (X'X)^{-1} X'\sigma_\varepsilon^2 I_n X(X'X)^{-1} \\ &= \sigma_\varepsilon^2 (X'X)^{-1} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\left(\hat{\boldsymbol{\beta}}\right) &= \mathrm{E}\left[\mathrm{Var}\left(\hat{\boldsymbol{\beta}}|\mathrm{X}\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\hat{\boldsymbol{\beta}}|\mathrm{X}\right]\right) \\ &= \mathrm{E}\left[\sigma_\varepsilon^2 (X'X)^{-1}\right] \\ &= \sigma_\varepsilon^2 \mathrm{E}\left[(X'X)^{-1}\right] \neq \sigma_\varepsilon^2 (X'X)^{-1} \end{aligned}$$

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5. Would your answer change if you made no assumption about a particular form of the variance-covariance matrix of the error term?
In general, $\mathrm{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|X\right] = \sigma_{\varepsilon}^{2}\Omega$. So

$$\mathrm{Var}\left(\hat{\boldsymbol{\beta}}|X\right) = \left(X'X\right)^{-1} X'\sigma_{\varepsilon}^{2}\Omega X \left(X'X\right)^{-1} \neq \sigma_{\varepsilon}^{2}\left(X'X\right)^{-1}$$