Econometrics, Michaelmas Term 2015, Quiz 2

Exercise 1. Suppose $\hat{\theta}$ is an estimator of some parameter θ (for example sample mean $\bar{Y}$ is an estimator of the population mean μ). Explain what it means for an estimator to be unbiased, consistent and efficient.

$$\begin{aligned} \text{Unbiased :} \quad & \mathrm{E}\left[\hat{\theta}\right] = \theta \\ \text{Consistent:} \quad & \mathrm{P}\left(|\hat{\theta} - \theta| > \varepsilon\right) \to 0, \text{ for any } \varepsilon > 0, \text{ as } n \to \infty \\ \text{Efficient:} \quad & \mathrm{Var}\left(\hat{\theta}\right) < \mathrm{Var}\left(\hat{\theta}'\right), \text{ for any } \hat{\theta}' \neq \hat{\theta} \end{aligned}$$

Exercise 2. List three conditions under which the Gauss-Markov Theorem for a *simple regression* (i.e. one regressor) is satisfied. Give formulas and provide explanations.

1. $\mathrm{E}\left[u_i | x_1, \ldots x_n\right] = 0$ for all i. The residual are independent from the regressors.
2. $\mathrm{Var}\left(u_i | x_1, \ldots, x_n\right) = \sigma_u^2$, where $0 < \sigma_u^2 < \infty$. Residuals are conditionally homoscedastic.
3. $\mathrm{E}\left[u_i u_j | x_1, \ldots, x_n\right] = 0$ for all $i \neq j$. Residuals are conditionally serially uncorrelated (nonautocorrelated).

Exercise 3. In a linear regression $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$, explain what it means for the coefficient estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ to be BLUE?

- Best (most efficient): estimates have the smallest variance among all unbiased linear estimates.
- Linear: estimates are a linear function of Y_i: $\hat{\beta}_1 = \sum_i^n a_i Y_i$. a_i can depend on x_i but not on y_i.
- (Conditionally) Unbiased: $\mathrm{E}\left[\hat{\beta}_1 | x_1, \ldots x_n\right] = \beta_1$.

Exercise 4. In a multiple regression $Y = X\beta + U$, the expression for the OLS estimator of β in matrix notation is $\hat{\beta} = \left(X'X\right)^{-1}\left(X'Y\right)$. Substitute the original regression model in this expression to obtain a relationship between $\hat{\beta}$ and β.

$$\begin{aligned} \hat{\beta} &= \left(X'X\right)^{-1}\left(X'Y\right) \\ &= \left(X'X\right)^{-1}\left(X'\left(X\beta + U\right)\right) \\ &= \left(X'X\right)^{-1}\left(X'X\right)\beta + \left(X'X\right)^{-1} X'U \\ &= \beta + \left(X'X\right)^{-1} X'U \end{aligned}$$

Exercise 5. Using your answer to Exercise 4, show that in a multiple regression $Y = X\beta + U$, the OLS estimator of β is unbiased. State explicitly which assumption of the Gauss-Markov Theorem is required for this to hold.

$$\begin{aligned} \mathrm{E}\left[\hat{\beta}\right] &= \\ \text{(Law of Iterarated Expectations)} \quad &= \mathrm{E}\left[\mathrm{E}\left[\hat{\beta}|X\right]\right] \\ \text{(Expression for the OLS estimator)} \quad &= \mathrm{E}\left[\mathrm{E}\left[\beta + \left(X'X\right)^{-1}\left(X'U\right)|X\right]\right] \\ \text{(Linearity of expectation operator)} \quad &= \mathrm{E}\left[\beta + \left(X'X\right)^{-1} X'\mathrm{E}\left[U|X\right]\right] \\ \text{(Independene of residuals and regressors)} \quad &= \beta \end{aligned}$$